

Error Analysis of Lagrange Interpolation on Tetrahedrons *

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Abstract. This paper describes the analysis of Lagrange interpolation errors on tetrahedrons. In many textbooks, the error analysis of Lagrange interpolation is conducted under geometric assumptions such as shape regularity or the (generalized) maximum angle condition. In this paper, we present a new estimation in which the error is bounded in terms of the diameter and projected circumradius of the tetrahedron. It should be emphasized that we do not impose any geometric restrictions on the tetrahedron itself.

Keywords. Lagrange interpolation, tetrahedrons, projected circumradius, finite elements

AMS subject classifications. 65D05, 65N30

1 Introduction

Lagrange interpolation on tetrahedrons and the associated error estimates are important subjects in numerical analysis. In particular, they are crucial in the error analysis of finite element methods. Throughout this paper, $K \subset \mathbb{R}^3$ denotes a tetrahedron with vertices \mathbf{x}_i , $i = 1, \dots, 4$, and tetrahedrons are assumed to be closed sets. Let λ_i be the barycentric coordinates of a tetrahedron with respect to \mathbf{x}_i . By definition, we have $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^4 \lambda_i = 1$. Let \mathbb{N}_0 be the set of nonnegative integers, and $\gamma = (a_1, \dots, a_4) \in \mathbb{N}_0^4$ be a multi-index. Let k be a positive integer. If $|\gamma| := \sum_{i=1}^4 a_i = k$, then $\gamma/k := (a_1/k, \dots, a_4/k)$ can be regarded as a barycentric coordinate in K . The set $\Sigma^k(K)$ of

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points on K is defined by

$$\Sigma^k(K) := \left\{ \frac{\gamma}{k} \in K \mid |\gamma| = k, \gamma \in \mathbb{N}_0^4 \right\}.$$

For k , \mathcal{P}_k is the set of all polynomials of three variables whose degree is at most k . For a continuous function $v \in C^0(K)$, the Lagrange interpolation $\mathcal{I}_K^k v \in \mathcal{P}_k$ of degree k is defined as

$$v(\mathbf{x}) = (\mathcal{I}_K^k v)(\mathbf{x}), \quad \forall \mathbf{x} \in \Sigma^k(K).$$

We attempt to obtain an upper bound of the error $|v - \mathcal{I}_K^k v|_{m,p,K}$ for integers $0 \leq m \leq k$, where $|\cdot|_{m,p,K}$ is the usual Sobolev semi-norm.

For now, let K be a d -simplicial element ($d = 2, 3$). Let $h_K := \text{diam}(K)$ and ρ_K be the diameter of its inscribed sphere. For the error analysis of Lagrange interpolation on simplicial elements, many textbooks on finite element methods, such as those by Ciarlet [6], Brenner and Scott [3], and Ern and Guermond [8], explain the following theorem.

Theorem 1 (Shape-regularity) *Let $\sigma > 0$ be a constant. If $h_K/\rho_K \leq \sigma$, then there exists a constant $C = C(\sigma)$ independent of h_K such that*

$$|v - \mathcal{I}_K^1 v|_{1,2,K} \leq C h_K |v|_{2,2,K}, \quad \forall v \in H^2(K). \quad (1)$$

Note that (1) does not hold if $d \geq 4$.

For the case of triangles, shape-regularity is equivalent to the *minimum angle condition* [18]. It is known that shape-regularity is not an optimal condition on the geometry of triangles. If the maximum angle of a triangle is less than a fixed constant $\theta_1 < \pi$, then the estimation (1) holds with a constant $C = C(\theta_1)$. This condition is known as the *maximum angle condition* [2, 9].

The present authors recently reported an error estimation in terms of the circumradius of a triangle [10, 12, 13]. Let R_K be the circumradius of a triangle K .

Theorem 2 (Circumradius estimates) *Let K be an arbitrary triangle. Let $1 \leq p \leq \infty$, and k, m be integers such that $k \geq 1$ and $0 \leq m \leq k$. Then, for the k th-order Lagrange interpolation \mathcal{I}_K^k on K , the estimation*

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leq C \left(\frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K} = C R_K^m h_K^{k+1-2m} |v|_{k+1,p,K} \quad (2)$$

holds for any $v \in W^{k+1,p}(K)$, where the constant $C = C(k, m, p)$ is independent of the geometry of K .

Note that the circumradius estimation (2) is closely related to the definition of surface area [11].

The aim of this paper is to derive a similar error estimation to Theorem 2 for tetrahedrons under no specific geometric restrictions.

To extend the circumradius estimation (2) to tetrahedrons, an immediate idea is to replace the circumradius of a triangle with the radius of the circumsphere of a tetrahedron. However, this idea can be immediately rejected by considering the tetrahedron K with vertices $\mathbf{x}_1 = (h, 0, 0)^\top$, $\mathbf{x}_2 = (-h, 0, 0)^\top$, $\mathbf{x}_3 = (0, -h, h^\alpha)^\top$, and $\mathbf{x}_4 = (0, h, h^\alpha)^\top$ with $h > 0$ and $\alpha > 0$. Setting $v_1(x, y, z) := x^2 - h^2 + h^{2-\alpha}z$, we have that $\mathcal{I}_K^1 v_1 \equiv 0$, and a simple computation yields $|v_1 - \mathcal{I}_K^1 v_1|_{1,\infty,K} = |v_1|_{1,\infty,K} \geq h^{2-\alpha}$ and $|v_1|_{2,\infty,K} = 2$. Hence, if $\alpha > 2$, an inequality such as (2) does not hold for tetrahedrons, although the radius of the circumsphere of the above K converges to 0 as $h \rightarrow 0$. Tetrahedrons such as the above K are called *slivers* [5].

Thus, we introduce the projected circumradius, also denoted by R_K , of a tetrahedron K in Section 2.5. We then obtain an error estimation for tetrahedrons that is fundamentally similar to (2) in Theorem 9.

In Section 2, we state some preliminary results used in this paper. In particular, we define the standard position and the projected circumradius for tetrahedrons. In Section 3, we recall definitions related to the quotient differences of functions with multi-variables. We then reconfirm the Squeezing Theorem for tetrahedrons. In Section 4, we obtain the error estimation of Lagrange interpolation in terms of the singular values of a linear transformation. In Section 5, we present a geometric interpretation of the singular values of the linear transformation, and finally obtain the main theorem.

2 Preliminaries

2.1 Notation

Let $d \geq 1$ be a positive integer and \mathbb{R}^d be the d -dimensional Euclidean space. We denote the Euclidean norm of $\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$ by $|\mathbf{x}|$. We always regard $\mathbf{x} \in \mathbb{R}^d$ as a column vector. For a matrix A and $\mathbf{x} \in \mathbb{R}^d$, A^\top and \mathbf{x}^\top denote their transpositions.

For $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}_0^d$, the multi-index ∂^δ of partial differentiation (in the sense of the distribution) is defined by

$$\partial^\delta = \partial_{\mathbf{x}}^\delta := \frac{\partial^{|\delta|}}{\partial x_1^{\delta_1} \dots \partial x_d^{\delta_d}}, \quad |\delta| := \delta_1 + \dots + \delta_d.$$

Let $\Omega \subset \mathbb{R}^d$ be a (bounded) domain. The usual Lebesgue space is denoted by $L^p(\Omega)$ for $1 \leq p \leq \infty$. For a positive integer k , the Sobolev space $W^{k,p}(\Omega)$ is defined by $W^{k,p}(\Omega) := \{v \in L^p(\Omega) \mid \partial^\delta v \in L^p(\Omega), |\delta| \leq k\}$. For $1 \leq p < \infty$, the norm and semi-norm of $W^{k,p}(\Omega)$ are defined by

$$|v|_{k,p,\Omega} := \left(\sum_{|\delta|=k} |\partial^\delta v|_{0,p,\Omega}^p \right)^{1/p}, \quad \|v\|_{k,p,\Omega} := \left(\sum_{0 \leq m \leq k} |v|_{m,p,\Omega}^p \right)^{1/p}$$

and $|v|_{k,\infty,\Omega} := \max_{|\delta|=k} \left\{ \text{ess sup}_{\mathbf{x} \in \Omega} |\partial^\delta v(\mathbf{x})| \right\}$, $\|v\|_{k,\infty,\Omega} := \max_{0 \leq m \leq k} \{|v|_{m,\infty,\Omega}\}$.

2.2 The imbedding theorem

Let $1 < p \leq \infty$. From Sobolev's imbedding theorem and Morry's inequality, we have the continuous imbeddings

$$\begin{aligned} W^{2,p}(K) &\subset C^{1,1-3/p}(K), \quad p > 3, \\ W^{2,3}(K) &\subset W^{1,q}(K) \subset C^{0,1-3/q}(K), \quad \forall q > 3, \\ W^{2,p}(K) &\subset W^{1,3p/(3-p)}(K) \subset C^{0,2-3/p}(K), \quad \frac{3}{2} < p < 3, \\ W^{3,3/2}(K) &\subset W^{2,3}(K) \subset W^{1,q}(K) \subset C^{0,1-3/q}(K), \quad \forall q > 3, \\ W^{3,p}(K) &\subset W^{2,3p/(3-p)}(K) \subset W^{1,3p/(3-2p)}(K) \subset C^{0,3-3/p}(K), \quad 1 < p < \frac{3}{2}. \end{aligned}$$

Although Morry's inequality may not be applied, the continuous imbedding $W^{3,1}(K) \subset C^0(K)$ still holds. For the imbedding theorem, see [1], [4], and [15]. In the following, we assume that p is such that the imbedding $W^{k+1,p}(K) \subset C^0(K)$ holds, that is,

$$1 \leq p \leq \infty, \quad \text{if } k+1 \geq 3 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \quad \text{if } k+1 = 2.$$

2.3 The reference tetrahedrons

Let \hat{K} and \tilde{K} be tetrahedrons that have the following vertices (see Figure 1):

$$\begin{aligned} \hat{K} &\text{ has the vertices } (0,0,0)^\top, (1,0,0)^\top, (0,1,0)^\top, (0,0,1)^\top, \\ \tilde{K} &\text{ has the vertices } (0,0,0)^\top, (1,0,0)^\top, (1,1,0)^\top, (0,0,1)^\top. \end{aligned}$$

These tetrahedrons are called the **reference tetrahedrons**. In this paper, we denote

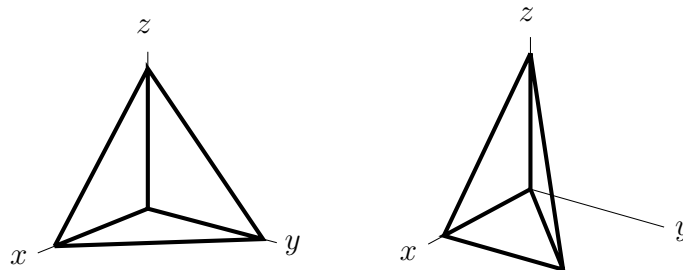


Figure 1: The reference tetrahedrons \hat{K} and \tilde{K} .

the reference tetrahedrons by \mathbf{K} , that is, \mathbf{K} is either of $\{\hat{K}, \tilde{K}\}$.

2.4 Standard position of tetrahedrons

When considering tetrahedrons, it is convenient to define their “standard coordinates.” Take any tetrahedron K with vertices \mathbf{x}_i , $i = 1, \dots, 4$. The facet B with vertices \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{x}_3 is regarded as the base of K . Let α and β , $0 < \beta \leq \alpha$, be the longest and shortest lengths of the edges of B . Without loss of generality, we assume that $\mathbf{x}_1\mathbf{x}_2$ is the longest edge of B ; $|\mathbf{x}_1 - \mathbf{x}_2| = \alpha$. Consider cutting \mathbb{R}^3 with the plane that contains the midpoint of the edge $\mathbf{x}_1\mathbf{x}_2$ and is perpendicular to the vector $\mathbf{x}_1 - \mathbf{x}_2$. Then, there exist two cases: (i) \mathbf{x}_3 and \mathbf{x}_4 belong to the same half-space, or (ii) \mathbf{x}_3 and \mathbf{x}_4 belong to different half-spaces. Let $\gamma := |\mathbf{x}_1 - \mathbf{x}_4|$. Under appropriate rotation, translation, and reflection operations, these situations can be written using the parameters

$$\begin{cases} 0 < \beta \leq \alpha, & 0 < \gamma, & s_1^2 + t_1^2 = 1, & s_1 > 0, & t_1 > 0, & \beta s_1 \leq \frac{\alpha}{2}, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, & t_2 > 0, & \gamma s_{21} \leq \frac{\alpha}{2}, \end{cases} \quad (3)$$

as

$$\mathbf{x}_1 = (0, 0, 0)^\top, \quad \mathbf{x}_2 = (\alpha, 0, 0)^\top, \quad \mathbf{x}_4 = (\gamma s_{21}, \gamma s_{22}, \gamma t_2)^\top, \quad (4a)$$

$$\begin{cases} \mathbf{x}_3 = (\beta s_1, \beta t_1, 0)^\top & \text{for the case (i)} \\ \mathbf{x}_3 = (\alpha - \beta s_1, \beta t_1, 0)^\top & \text{for the case (ii)} \end{cases}. \quad (4b)$$

We refer to the coordinates in (4) as the *standard position* of a tetrahedron. In the following, we sometimes write $h_B := \alpha$. Let R_B be the circumradius of B .

2.5 The projected circumradius of tetrahedrons

Suppose that a tetrahedron K is at the standard position. Let $\theta \in \mathbb{R}$ be such that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Let δ_θ be the linear transformation defined by the matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

That is, δ_θ is a composite transformation of rotation about the z -axis with angle θ and projection to the xz -plane. The image $\delta_\theta(K)$ is a triangle on the xz -plane. Let R_θ be the circumradius of $\delta_\theta(K)$. Define

$$R_P := \max_{\theta \in [-\pi/2, \pi/2]} R_\theta.$$

The *projected circumradius* R_K of a tetrahedron K is defined by

$$R_K := \min_B \frac{R_B R_P}{h_B}, \quad (5)$$

where the minimum is taken over all the facets of K .

2.6 The setting of error estimation

We define the set $\mathcal{T}_p^k(K) \subset W^{k+1,p}(K)$ by

$$\mathcal{T}_p^k(K) := \left\{ v \in W^{k+1,p}(K) \mid v(\mathbf{x}) = 0, \forall \mathbf{x} \in \Sigma^k(K) \right\}.$$

For Lagrange interpolation $\mathcal{I}_K^k(v)$, it is clear from the definition that

$$v - \mathcal{I}_K^k v \in \mathcal{T}_p^k(K), \quad \forall v \in W^{k+1,p}(K).$$

For an integer m such that $0 \leq m \leq k$, $B_p^{m,k}(K)$ is defined by

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$

Note that we have

$$B_p^{m,k}(K) = \inf \left\{ C; |v - \mathcal{I}_K^k v|_{m,p,K} \leq C |v|_{k+1,p,K}, \forall v \in W^{k+1,p}(K) \right\},$$

that is, $B_p^{m,k}(K)$ is the *best* constant C for the error estimation

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leq C |v|_{k+1,p,K}, \quad \forall v \in W^{k+1,p}(K).$$

Therefore, we try to obtain an upper bound of $B_p^{m,k}(K)$ in terms of geometric quantities of K .

3 The Squeezing Theorem

Let $a, b \in \mathbb{R}$ be such that $0 < a \leq 1$ and $0 < b$. We then define the *squeezing map* $sq_1^{ab} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$sq_1^{ab}(x, y, z) := (x, ay, bz)^\top, \quad (x, y, z)^\top \in \mathbb{R}^3.$$

Let $K_{ab} := sq_1^{ab}(\mathbf{K})$. In this section, we will prove the following theorem.

Theorem 3 *Let k be a positive integer and $1 \leq p \leq \infty$. There exists a constant $C_{k,m,p}$ independent of a and b such that, for $m = 0, \dots, k$,*

$$B_p^{m,k}(K_{ab}) := \sup_{v \in \mathcal{T}_p^k(K_{ab})} \frac{|v|_{m,p,K_{ab}}}{|v|_{k+1,p,K_{ab}}} \leq \max\{1, b^{k+1-m}\} C_{k,m,p},$$

where p is taken such that

$$\begin{cases} 2 < p \leq \infty & \text{if } k - m = 0, \\ \frac{3}{2} < p \leq \infty & \text{if } k = 1, m = 0, \\ 1 \leq p \leq \infty & \text{if } k \geq 2 \text{ and } k - m \geq 1. \end{cases} \quad (6)$$

Although the proof of Theorem 3 is very similar to that of [13, Theorem 1.3], we provide a sketch here for readers' convenience. Note that the restriction $2 < p \leq \infty$ for the case $k = m$ comes from the continuity of the trace operator $t : W^{1,p}(\mathbf{K}) \ni v \mapsto v|_S \in L^1(S)$, where $S \subset \mathbf{K}$ is a non-degenerate segment (see [13, Section 3]). Using the counter-example given by Shenk [17], we find that this restriction cannot be improved.

To prove Theorem 3, we recall the definitions of difference quotients of multi-variable functions, the rectangular parallelepiped \square_γ^δ defined by the lattice points \mathbf{x}_γ , $\Delta^\delta \mathbf{x}_\gamma$ in \mathbf{K} , and the integral $\int_{\square_\gamma^\delta} v$ introduced in [13, Section 2]. The set $\Xi_p^{\delta,k} \subset W^{k+1-|\delta|,p}(\mathbf{K})$ is then defined by

$$\Xi_p^{\delta,k} := \left\{ v \in W^{k+1-|\delta|,p}(\mathbf{K}) \mid \int_{\square_\gamma^\delta} v = 0, \quad \forall \square_\gamma^\delta \subset \mathbf{K} \right\}.$$

Note that $u \in \mathcal{T}_p^k(\mathbf{K})$ implies $\partial^\delta u \in \Xi_p^{\delta,k}$.

Lemma 4 *We have $\Xi_p^{\delta,k} \cap \mathcal{P}_{k-|\delta|} = \{0\}$. That is, if $q \in \mathcal{P}_{k-|\delta|}$ belongs to $\Xi_p^{\delta,k}$, then $q = 0$.*

Proof. Note that $\dim \mathcal{P}_{k-|\delta|} = \#\{\square_{l_p}^\delta \subset \mathbf{K}\}$. For example, if $k = 4$ and $|\delta| = 3$, then $\dim \mathcal{P}_1 = 4$. This corresponds to the fact that, in \mathbf{K} , there are four cubes of size $1/4$ for $\delta = (1, 1, 1)$ and there are four rectangles of size $1/2 \times 1/4$ for $\delta = (1, 2, 0)$. All their vertices (corners) belong to $\Sigma^4(\mathbf{K})$ (see Figure 2). Now, suppose that $v \in \mathcal{P}_{k-|\delta|}$ satisfies $\int_{\square_{l_p}^\delta} q = 0$ for all $\square_{l_p}^\delta \subset \mathbf{K}$. These conditions are linearly independent and determine $q = 0$ uniquely. \square

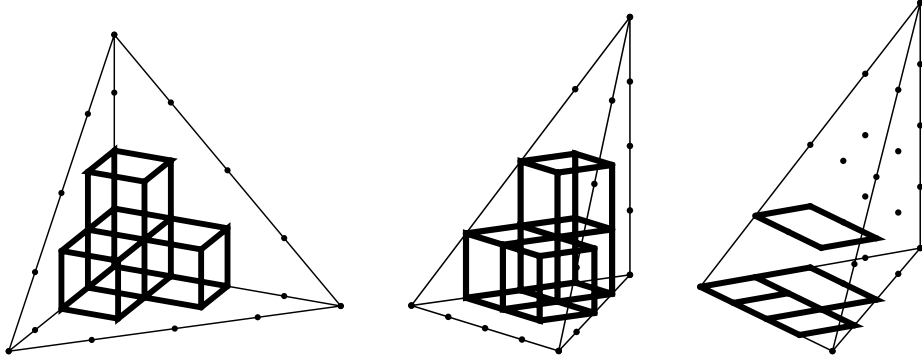


Figure 2: The four cubes and four rectangles in \mathbf{K} .

The constant $A_p^{\delta,k}$ is defined by

$$A_p^{\delta,k} := \sup_{v \in \Xi_p^{\delta,k}} \frac{|v|_{0,p,\mathbf{K}}}{|v|_{k+1-|\delta|,p,\mathbf{K}}}.$$

The following lemma is an extension of [2, Lemma 2.1].

Lemma 5 Let p be such that $2 < p \leq \infty$ if $k+1-|\delta| = 1$ or $1 \leq p \leq \infty$ if $k+1-|\delta| \geq 2$. We then have $A_p^{\delta,k} < \infty$.

Proof. See the proof of [13, Lemma 3.3]. \square

Proof of Theorem 3. First, let $1 \leq p < \infty$ and $1 \leq m \leq k$. For a multi-index $\gamma = (n_1, n_2, n_3) \in \mathbb{N}_0^3$ and a real $t \neq 0$, set $(a, b)^{\gamma t} := a^{n_2 t} b^{n_3 t}$. Take an arbitrary $v \in \mathcal{T}_p^k(K_{ab})$ and pull it back to $u := v \circ sq_1^{ab} \in \mathcal{T}_p^k(\mathbf{K})$. We then have

$$\begin{aligned} \frac{|v|_{m,p,K_{ab}}^p}{|v|_{k+1,p,K_{ab}}^p} &= \frac{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} |\partial^\gamma u|_{0,p,\mathbf{K}}^p}{\sum_{|\delta|=k+1} \frac{(k+1)!}{\delta!} (a, b)^{-\delta p} |\partial^\delta u|_{0,p,\mathbf{K}}^p} \\ &= \frac{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} |\partial^\gamma u|_{0,p,\mathbf{K}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} \left(\sum_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta! (a,b)^{\eta p}} |\partial^\eta (\partial^\gamma u)|_{0,p,\mathbf{K}}^p \right)} \\ &\leq \frac{\max\{1, b^{(k+1-m)p}\} \sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} |\partial^\gamma u|_{0,p,\mathbf{K}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} \left(\sum_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta!} |\partial^\eta (\partial^\gamma u)|_{0,p,\mathbf{K}}^p \right)} \\ &= \max\{1, b^{(k+1-m)p}\} \frac{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} |\partial^\gamma u|_{0,p,\mathbf{K}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (a, b)^{-\gamma p} |\partial^\gamma u|_{k+1-m,p,\mathbf{K}}^p} \\ &\leq \max\{1, b^{(k+1-m)p}\} \max_{|\gamma|=m} (A_p^{\gamma,k})^p = (\max\{1, b^{k+1-m}\} C_{k,m,p})^p, \end{aligned}$$

where $C_{k,m,p} := \max_{|\gamma|=m} A_p^{\gamma,k}$. Proofs for the cases $1 \leq p < \infty$ with $m = 0$ and $p = \infty$ with $0 \leq m \leq k$ are very similar. See the proof of [13, Theorem 1.3]. \square

We now generalize the squeezing map. Let α, β , and $\gamma \in \mathbb{R}$ be such that $0 < \beta \leq \alpha$ and $0 < \gamma$. We then define the *squeezing map* $sq_2^{\alpha\beta\gamma} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$sq_2^{\alpha\beta\gamma}(x, y, z) := (\alpha x, \beta y, \gamma z)^\top, \quad (x, y, z)^\top \in \mathbb{R}^3.$$

Let $K_{\alpha\beta\gamma} := sq_2^{\alpha\beta\gamma}(\mathbf{K})$. Note that $G_\alpha := sq_2^{\alpha\alpha\alpha}$ is a similar transformation. Let $\Omega \subset \mathbb{R}^3$ be a domain and an arbitrary function $v \in W^{k,p}(\Omega)$ be pulled-back to $u := v \circ G_\alpha \in W^{k,p}(\Omega_\alpha)$ with $\Omega_\alpha := G_{1/\alpha}(\Omega)$. It is straightforward to check that

$$|v|_{k,p,\Omega} = \alpha^{3/p-k} |u|_{k,p,\Omega_\alpha}.$$

Now, take an arbitrary function $v \in \mathcal{T}_p^k(K_{\alpha\beta\gamma})$ and define $u := v \circ sq_2^{\alpha\beta\gamma}$. Then, $u = u_1 \circ sq_1^{\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}}$ with $u_1 := v \circ G_\alpha$, because $sq_2^{\alpha\beta\gamma} = G_\alpha \circ sq_1^{\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}}$. Therefore, it follows from Theorem 3 that

$$\begin{aligned} \frac{|v|_{m,p,K_{\alpha\beta\gamma}}}{|v|_{k+1,p,K_{\alpha\beta\gamma}}} &= \alpha^{k+1-m} \frac{|u_1|_{m,p,K_{\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}}}}{|u_1|_{k+1,p,K_{\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}}}} \leq \alpha^{k+1-m} \max \left\{ 1, \left(\frac{\gamma}{\alpha} \right)^{k+1-m} \right\} C_{k,m,p} \\ &= (\max\{\alpha, \gamma\})^{k+1-m} C_{k,m,p}. \end{aligned}$$

Hence, we have derived the following corollary.

Corollary 6 *Let α , β , and $\gamma \in \mathbb{R}$ be such that $0 < \beta \leq \alpha$ and $0 < \gamma$. Let $K_{\alpha\beta\gamma} := sq_2^{\alpha\beta\gamma}(\mathbf{K})$. Assume that $k \geq 1$, $0 \leq m \leq k$, and p is taken as (6). We then have*

$$B_p^{m,k}(K_{\alpha\beta\gamma}) := \sup_{v \in \mathcal{T}_p^k(K_{\alpha\beta\gamma})} \frac{|v|_{m,p,K_{\alpha\beta\gamma}}}{|v|_{k+1,p,K_{\alpha\beta\gamma}}} \leq (\max\{\alpha, \gamma\})^{k+1-m} C_{k,m,p}.$$

4 Error estimates of Lagrange interpolation on general tetrahedrons

In this section, we obtain an error estimation for Lagrange interpolation on general tetrahedrons. To this end, we apply the method developed in [12].

Recall that an arbitrary tetrahedron K is written as (4) with parameters (3). First, we confirm that K is obtained from the reference tetrahedron \mathbf{K} by an affine linear transformation. Define the matrices $\hat{A}, \tilde{A}, G \in GL(3, \mathbb{R})$ by

$$\hat{A} := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \tilde{A} := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad G := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

We then have $K = \hat{A}G(\hat{K})$ for case (i) or $K = \tilde{A}G(\tilde{K})$ for case (ii), that is, $K = \hat{A}(K_{\alpha\beta\gamma})$ or $K = \tilde{A}(K_{\alpha\beta\gamma})$. Note that \hat{A} and \tilde{A} can be decomposed as $\hat{A} = X\hat{Y}$ and $\tilde{A} = X\tilde{Y}$ with

$$X := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \hat{Y} := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{Y} := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. We consider the singular values of \hat{A} , \tilde{A} , X , \hat{Y} , and \tilde{Y} . A straightforward computation yields

$$\det(X^\top X - \mu I) = (1 - \mu)(\mu^2 - 2\mu + t_2^2),$$

$$\det(\hat{Y}^\top \hat{Y} - \mu I) = \det(\tilde{Y}^\top \tilde{Y} - \mu I) = (1 - \mu)(\mu^2 - 2\mu + t_1^2),$$

the eigenvalues of which are $\mu = 1$, $1 \pm \sqrt{1 - t_i^2} = 1 \pm \mathbf{s}_i$, $i = 1, 2$, where $\mathbf{s}_1 := |s_1|$ and $\mathbf{s}_2 := \sqrt{s_{21}^2 + s_{22}^2}$. Therefore, for $\mathbf{a} \in \mathbb{R}^3$, we have

$$(1 - \mathbf{s}_2)|\mathbf{a}|^2 \leq |X\mathbf{a}|^2 \leq (1 + \mathbf{s}_2)^2|\mathbf{a}|^2,$$

$$(1 - \mathbf{s}_1)|\mathbf{a}|^2 \leq |Z\mathbf{a}|^2 \leq (1 + \mathbf{s}_1)|\mathbf{a}|^2, \quad Z = \hat{Y} \text{ or } Z = \tilde{Y},$$

$$\prod_{i=1}^2 (1 - \mathbf{s}_i)|\mathbf{a}|^2 \leq |V\mathbf{a}|^2 \leq \prod_{i=1}^2 (1 + \mathbf{s}_i)|\mathbf{a}|^2, \quad V = \hat{A} \text{ or } V = \tilde{A}.$$

Let $K := V(K_{\alpha\beta\gamma})$, where $V = \widehat{A}$ or $V = \widetilde{A}$. A function $v \in W^{r,p}(K)$ is pulled-back to a function $u \in W^{r,p}(K_{\alpha\beta\gamma})$ by $u(\mathbf{x}) := v(V\mathbf{x})$. Using inequality (2.1) in [12], we have

$$\frac{\prod_{i=1}^2 (1 - \mathbf{s}_i)^r}{t_1^{2r} t_2^{2r}} \sum_{|\gamma|=r} (\partial_{\mathbf{x}}^\gamma u)^2 \leq \sum_{|\gamma|=r} (\partial_{\mathbf{y}}^\gamma v)^2 \leq \frac{\prod_{i=1}^2 (1 + \mathbf{s}_i)^r}{t_1^{2r} t_2^{2r}} \sum_{|\gamma|=r} (\partial_{\mathbf{x}}^\gamma u)^r,$$

where $\mathbf{y} := V\mathbf{x}$. The fact that $\det A = t_1 t_2$ and inequalities (2.2), (2.3) in [12] then give

$$\begin{aligned} |v|_{m,p,K} &\leq 3^{m\tau(p)} \frac{\prod_{i=1}^2 (1 + \mathbf{s}_i)^{m/2}}{(t_1 t_2)^{m-1}} |u|_{m,p,K_{\alpha\beta\gamma}}, \\ |v|_{k+1,p,K} &\geq 3^{-(k+1)\tau(p)} \frac{\prod_{i=1}^2 (1 - \mathbf{s}_i)^{(k+1)/2}}{(t_1 t_2)^k} |u|_{k+1,p,K_{\alpha\beta\gamma}}, \\ \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} &\leq 3^{(k+1+m)\tau(p)} \frac{(t_1 t_2)^{k+1-m} \prod_{i=1}^2 (1 + \mathbf{s}_i)^{m/2} |u|_{m,p,K_{\alpha\beta\gamma}}}{\prod_{i=1}^2 (1 - \mathbf{s}_i)^{(k+1)/2} |u|_{k+1,p,K_{\alpha\beta\gamma}}} \\ &= 3^{(k+1+m)\tau(p)} \frac{\prod_{i=1}^2 (1 + \mathbf{s}_i)^{(k+1)/2}}{\prod_{i=1}^2 (1 - \mathbf{s}_i)^{m/2}} \frac{|u|_{m,p,K_{\alpha\beta\gamma}}}{|u|_{k+1,p,K_{\alpha\beta\gamma}}} \\ &\leq C \frac{(\max\{\alpha, \gamma\})^{k+1-m}}{\prod_{i=1}^2 (1 - \mathbf{s}_i)^{m/2}}, \end{aligned}$$

where

$$\tau(p) := \begin{cases} \frac{1}{p} - \frac{1}{2}, & 1 \leq p \leq 2 \\ \frac{1}{2} - \frac{1}{p}, & 2 \leq p \leq \infty \end{cases}, \quad C := 3^{(k+1+m)\tau(p)} 2^{k+1} C_{k,m,p}.$$

(See the inequalities in [12, p.496].) Note that the constant C depends only on k , m , and p . Hence, we have derived the following theorem:

Theorem 7 *Let K be an arbitrary tetrahedron with vertices given by (4) with the parameters in (3). Let k, m be integers with $k \geq 1$ and $0 \leq m \leq k$. Let p be taken as in Theorem 3 according to k and m . Then, we have*

$$B_p^{m,k}(K) := \sup_{v \in T_p^k} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} \leq C \frac{(\max\{\alpha, \gamma\})^{k+1-m}}{\prod_{i=1}^2 (1 - \mathbf{s}_i)^{m/2}},$$

where $C = C(k, m, p)$ is a constant independent of K .

5 A geometric interpretation and the main theorem

In this section, we consider the geometric meaning of the quantity $\prod_{i=1}^2 (1 - \mathbf{s}_i)^{-1/2}$ that appeared in Theorem 7. Recall that K is a tetrahedron with vertices given by (4) and

the parameters in (3). Recall also that B is the base of K with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. The circumradius R_B can then be written as

$$R_B = \frac{\sqrt{\alpha^2 - 2\alpha\beta s_1 + \beta^2}}{2t_1}.$$

Because

$$\begin{aligned} \alpha^2 - 2\alpha\beta s_1 + \beta^2 &= \frac{\alpha^2(2 - s_1^2)}{4} + \alpha \left(\frac{\alpha}{2} - \beta s_1 \right) + \left(\frac{\alpha s_1}{2} - \beta \right)^2 \\ &\geq \frac{\alpha^2(2 - s_1^2)}{4} \geq \frac{\alpha^2}{4}, \end{aligned}$$

we have

$$R_B \geq \frac{\alpha}{4t_1} = \frac{\alpha}{4\sqrt{1 - \mathbf{s}_1^2}} \geq \frac{h_B}{4\sqrt{2}\sqrt{1 - \mathbf{s}_1}}, \quad (7)$$

where we have used the definition $h_B = \alpha$.

Recall that R_P was defined in Section 2.5. We will show that there exists a constant C independent of K such that

$$R_P \geq C \frac{\max\{\alpha, \gamma\}}{\sqrt{1 - \mathbf{s}_2}}. \quad (8)$$

Let $\mathbf{x}_3 = (\eta, \xi, 0)^\top$, that is, $\xi = \beta t_1$ and $\eta = \beta s_1$ or $\eta = \alpha - \beta s_1$. From the assumption, we have $0 < \eta < \alpha$ and $\xi > 0$. Note that

$$\begin{aligned} \delta_\theta(\mathbf{x}_1) &= (0, 0, 0)^\top, \quad \delta_\theta(\mathbf{x}_2) = (\alpha \cos \theta, 0, 0)^\top, \\ \delta_\theta(\mathbf{x}_3) &= (\eta \cos \theta - \xi \sin \theta, 0, 0)^\top, \quad \delta_\theta(\mathbf{x}_4) = (\gamma(s_{21} \cos \theta - s_{22} \sin \theta, 0, t_2)^\top. \end{aligned}$$

Let $\underline{x} < \bar{x}$ be the x -coordinates of the end points of the base of $\delta_\theta(K)$. The assumptions given in (3) yield

$$\underline{x} \leq 0, \quad \bar{x} = \alpha \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad \underline{x} = 0, \quad \bar{x} \geq \alpha \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq 0.$$

Defining $w = w(\theta) := s_{21} \cos \theta - s_{22} \sin \theta$, R_θ is written as

$$R_\theta = \frac{1}{2\gamma t_2} \left((\bar{x} - \gamma w)^2 + \gamma^2 t_2^2 \right)^{1/2} \left((\underline{x} - \gamma w)^2 + \gamma^2 t_2^2 \right)^{1/2}.$$

Take an arbitrary $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. Suppose that the inequality

$$\gamma w \leq \frac{\underline{x} + \bar{x}}{2} \quad (9)$$

holds and there exists a constant C_1 independent of θ such that

$$|\underline{x} - \gamma w| \geq C_1 \gamma \mathbf{s}_2. \quad (10)$$

We then have

$$\begin{aligned}
(\underline{x} - \gamma w)^2 + \gamma^2 t_2^2 &\geq C_1^2(1 - t_2^2) + \gamma^2 t_2^2 \geq \min\{1, C_1^2\}\gamma^2, \\
(\bar{x} - \gamma w)^2 + \gamma^2 t_2^2 &\geq \max\left\{(\underline{x} - \gamma w)^2 + \gamma^2 t_2^2, \frac{\alpha^2 \cos^2 \theta}{4} + \gamma^2 t_2^2\right\} \\
&\geq \max\{\min\{1, C_1^2\}\gamma^2, C_2^2 \alpha^2\} \\
&\geq \min\{C_1^2, C_2^2\} \max\{\alpha^2, \gamma^2\},
\end{aligned}$$

where $C_2 = C_2(\theta) := \frac{\cos \theta}{2}$. Here, we have used the fact that $\bar{x}/2 \geq (\bar{x} + \underline{x})/2 \geq \gamma w$ and $\bar{x} - \gamma w \geq \bar{x}/2 \geq (\alpha \cos \theta)/2$. Hence, setting

$$\frac{\min\{C_1, C_2\} \min\{1, C_1\}}{2\sqrt{2}} \geq \frac{\min\{C_1, \frac{1}{4}\} \min\{1, C_1\}}{2\sqrt{2}} =: C_3,$$

we obtain

$$R_P \geq R_\theta \geq \frac{\sqrt{2}C_3}{t_2} \max\{\alpha, \gamma\} = \frac{\sqrt{2}C_3 \max\{\alpha, \gamma\}}{\sqrt{1 - s_2^2}} \geq C_3 \frac{\max\{\alpha, \gamma\}}{\sqrt{1 - s_2^2}},$$

and the key inequality (8) is shown.

Fix φ such that

$$0 < \varphi < \frac{\pi}{6}, \quad \sin 2\varphi \tan 2\varphi \leq \frac{1}{6}.$$

In the following, we will show that, according to $\mathbf{x}_4 = (\gamma s_{21}, \gamma s_{22}, \gamma t_2)^\top$, we can take an appropriate $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ such that conditions (9) and (10) hold with $C_1 = \sin \varphi$.

Case 1. Suppose that $|s_{22}| \tan \varphi \leq |s_{21}|$.

In this case, we set $\theta = 0$ and have $\underline{x} = 0$, $\bar{x} = \alpha$, and $\gamma w = \gamma s_{21} \leq \alpha/2 = (\underline{x} + \bar{x})/2$ because of (3). Hence, (9) holds. For (10), we note that

$$\begin{aligned}
|\underline{x} - \gamma w| &= \gamma |s_{21}| = \gamma (s_{21}^2 \sin^2 \varphi + s_{21}^2 \cos^2 \varphi)^{1/2} \\
&\geq \gamma (s_{21}^2 \sin^2 \varphi + s_{22}^2 \sin^2 \varphi)^{1/2} = \gamma s_2 \sin \varphi,
\end{aligned}$$

and so (10) holds with $C_1 := \sin \varphi$.

Case 2: Suppose that $|s_{22}| \tan \varphi > |s_{21}|$ and $3\gamma s_{22} \tan 2\varphi \leq \alpha$.

In this case, set $\theta = -2\varphi$. We then have $\underline{x} = 0$ and $\bar{x} \geq \alpha \cos \theta = \alpha \cos 2\varphi$. If $s_{22} > 0$, then

$$\begin{aligned}
\gamma w &= \gamma(s_{21} \cos 2\varphi + s_{22} \sin 2\varphi) \leq \gamma(s_{22} \cos 2\varphi \tan \varphi + s_{22} \sin 2\varphi) \\
&= \frac{\gamma s_{22}}{2}(3 \sin 2\varphi - 2 \sin^2 \varphi \tan \varphi) \\
&\leq \frac{2\gamma s_{22}}{2} \sin 2\varphi \leq \frac{\alpha \cos 2\varphi}{2} \leq \frac{\underline{x} + \bar{x}}{2}.
\end{aligned}$$

If $s_{22} \leq 0$, then

$$\begin{aligned}\gamma w &= \gamma(s_{21} \cos 2\varphi + s_{22} \sin 2\varphi) \leq \gamma(-s_{22} \cos 2\varphi \tan \varphi + s_{22} \sin 2\varphi) \\ &= \gamma s_{22} \tan \varphi \leq 0 \leq \frac{\underline{x} + \bar{x}}{2}.\end{aligned}$$

Thus, in either case, (9) holds. For (10), we note that

$$\begin{aligned}|\underline{x} - \gamma w| &= \gamma |s_{21} \cos 2\varphi + s_{22} \sin 2\varphi| \geq \gamma (|s_{22}| \sin 2\varphi - |s_{21}| \cos 2\varphi) \\ &\geq \gamma (|s_{22}| \sin 2\varphi - |s_{22}| \cos 2\varphi \tan \varphi) = \gamma |s_{22}| \tan \varphi \\ &= \gamma (s_{22}^2 \sin^2 \varphi \tan^2 \varphi + s_{22}^2 \sin^2 \varphi)^{1/2} \\ &= \gamma (s_{21}^2 \sin^2 \varphi + s_{22}^2 \sin^2 \varphi)^{1/2} = \gamma \mathbf{s}_2 \sin \varphi,\end{aligned}$$

and so (10) holds with $C_1 := \sin \varphi$.

Case 3: Suppose that $|s_{22}| \tan \varphi > |s_{21}|$ and $3\gamma s_{22} \tan 2\varphi > \alpha$.

In this case, set $\theta = 2\varphi$. We then have

$$\underline{x} = \min\{\eta \cos 2\varphi - \xi \sin 2\varphi, 0\}, \quad \bar{x} = \alpha \cos 2\varphi.$$

If $\eta \cos 2\varphi - \xi \sin 2\varphi \leq 0$, then

$$\begin{aligned}\underline{x} &= \eta \cos 2\varphi - \xi \sin 2\varphi = \alpha \cos 2\varphi + (\eta - \alpha) \cos 2\varphi - \xi \sin 2\varphi \\ &\geq \alpha \cos 2\varphi - ((\eta - \alpha)^2 + \xi^2)^{1/2} \geq \alpha \cos 2\varphi - \alpha = -2\alpha \sin^2 \varphi,\end{aligned}$$

because the lengths of all edges of the base are less than α . Even if $\eta \cos 2\varphi - \xi \sin 2\varphi \geq 0 = \underline{x}$, the above inequality obviously holds. Because

$$\begin{aligned}\gamma w &= \gamma(s_{21} \cos 2\varphi - s_{22} \sin 2\varphi) \\ &\leq \gamma(s_{22} \cos 2\varphi \tan \varphi - s_{22} \sin 2\varphi) = -\gamma s_{22} \tan \varphi,\end{aligned}$$

we have

$$\begin{aligned}\underline{x} - \gamma w &\geq -2\alpha \sin^2 \varphi + \gamma s_{22} \tan \varphi \\ &\geq -2\alpha \sin^2 \varphi + \frac{\alpha \tan \varphi}{6 \tan 2\varphi} + \frac{\gamma}{2} s_{22} \tan \varphi \\ &= \alpha \left(\frac{1}{6} - \sin 2\varphi \tan 2\varphi \right) \frac{\tan \varphi}{\tan 2\varphi} + \frac{\gamma}{2} s_{22} \tan \varphi \geq \frac{\gamma}{2} s_{22} \tan \varphi > 0.\end{aligned}$$

Therefore,

$$\gamma w < \underline{x} < \frac{\underline{x} + \bar{x}}{2},$$

and (9) holds. For (10), we note that

$$\begin{aligned}
|\underline{x} - \gamma w| &\geq \frac{\gamma}{2} s_{22} \tan \varphi \\
&= \gamma (s_{22}^2 \sin^2 \varphi \tan^2 \varphi + s_{22}^2 \sin^2 \varphi)^{1/2} \\
&\geq \gamma (s_{21}^2 \sin^2 \varphi + s_{22}^2 \sin^2 \varphi)^{1/2} = \gamma \mathbf{s}_2 \sin \varphi,
\end{aligned}$$

and so (10) holds with $C_1 := \sin \varphi$.

Using inequalities (7) and (8), we have shown the following lemma.

Lemma 8 *Let K be a tetrahedron with vertices given by (4) and the parameters in (3). We then have*

$$\prod_{i=1}^2 (1 - \mathbf{s}_i)^{-1/2} \leq C \frac{R_B R_P}{h_B \max\{\alpha, \gamma\}},$$

where C is a constant independent of K .

Take a facet B so that $R_B R_P / h_B = R_K$. Combining Theorem 7 and Lemma 8 with the projected circumradius R_K , we have

$$\begin{aligned}
B_p^{m,k}(K) &:= \sup_{v \in \mathcal{T}_p^k} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} \leq C \frac{(\max\{\alpha, \gamma\})^{k+1-m}}{\prod_{i=1}^2 (1 - \mathbf{s}_i)^{m/2}} \\
&\leq C \left(\frac{R_B R_P}{h_B} \right)^m (\max\{\alpha, \gamma\})^{k+1-2m} \\
&\leq C R_K^m h_K^{k+1-2m},
\end{aligned}$$

where the constant $C = C(k, m, p)$ is independent of K with vertices (4). Note that Sobolev (semi-)norms are affected by rotation up to a constant. Therefore, we have derived the following result, which is the main theorem of this paper.

Theorem 9 *Let K be an arbitrary tetrahedron. Let $h_K := \text{diam} K$ and R_K be the projected circumradius of K defined by (5). Assume that k, m are integers with $k \geq 1$, $0 \leq m \leq k$, and p ($1 \leq p \leq \infty$) is taken as in (6). Then, for arbitrary $v \in W^{k+1,p}(K)$, there exists a constant $C = C(k, m, p)$ independent of K such that*

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leq C R_K^m h_K^{k+1-2m} |v|_{k+1,p,K} = C \left(\frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K}. \quad (11)$$

Note that the error estimation (11) is exactly same as (2).

6 Concluding remarks

In this final section, we present some remarks on the newly obtained error estimation.

(1) For finite element error analysis, the most important case is $p = 2$ and $m = 1$. In this case, k should be greater than or equal to 2 because of the restriction in (6). In this case, the main estimation is

$$|v - \mathcal{I}_K^k v|_{1,p,K} \leq C R_K h_K^{k-1} |v|_{k+1,p,K} = C \left(\frac{R_K}{h_K} \right) h_K^k |v|_{k+1,p,K}.$$

Let $\Omega \subset \mathbb{R}^3$ be a bounded polygonal domain. Suppose that we compute a numerical solution of the Poisson equation

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

by the finite element method with conforming simplicial elements. One difficulty we may encounter in the numerical computation is the bad triangulation of Ω with many slivers [5]. Consider the typical sliver K mentioned in Section 1, whose vertices are $(\pm h, 0, 0)^\top$ and $(0, \pm h, h^\alpha)^\top$. We can see that $R_K = \mathcal{O}(h_K^{2-\alpha})$ and $R_K h_K^{k-1} = \mathcal{O}(h_K^{k+1-\alpha})$. Thus, if we use a k th-order conforming Lagrange finite element method with triangulation that contains many slivers like K , the theoretical convergence rate may be $\mathcal{O}(h^{k+1-\alpha})$. This is worse than the expected rate $\mathcal{O}(h^k)$, but we can still expect convergence if $k + 1 - \alpha > 0$. Therefore, bad triangulation with many slivers can be remedied by using higher-order Lagrange elements.

(2) Let K be a tetrahedron with the inscribed sphere S_K . If the regularity assumption is imposed on K , there exists a constant σ such that $h_K/\rho_K \leq \sigma$, where $\rho_K := \text{diam} S_K$. Suppose that K is at the standard position. Recall δ_θ , the projection introduced in Section 2.5. It is clear that $\delta_\theta(S_K) \subset \delta_\theta(K)$ and $\text{diam} \delta_\theta(S_K) = \rho_K$. Therefore, recalling that R_P is the maximum value of the circumradius of $\delta_\theta(K)$, there exists a constant C such that $R_P \leq C h_K$. The regularity assumption implies the maximum angle condition, and as was pointed out in [12, Section 4.1], the maximum angle condition of B implies the boundedness of R_B/h_B . Thus, Theorem 9 is an extension of the standard estimation (1).

(3) For tetrahedrons, a (generalized) maximum angle condition was given by Křížek [14], [7]. A slightly more general condition has been stated by Jamet [9]. The authors conjecture that Theorem 9 is an extension of these results.

(4) In Theorem 7, we obtained an error estimation for Lagrange interpolation on tetrahedrons in terms of the singular values $\prod_{i=1}^2 (1 - \mathbf{s}_i)^{-1/2}$ of the linear transformation. In Theorem 9, we showed that the projected circumradius is a geometric interpretation of the singular values. The authors, however, are not completely sure whether the projected circumradius is the *best* interpretation.

Further research on the geometry of tetrahedrons is required to ascertain the geometric properties of the singular values $\prod_{i=1}^2(1 - \mathbf{s}_i)^{-1/2}$ and the projected circumradius, and the relationship between these singular values and prior results.

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